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Renormalisation-group theory of the phase transition in ultrametric spin glasses

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Abstract. Spin-glass systems possessing the hierarchical tree-like structure of thermodynamically relevant metastable states are considered. A renormalisation-group scheme for the calculation of the partition function is proposed, and a systematic method for the calculation of the renormalised Hamiltonian with increasing scale in the space of states is derived. The renormalisation-group equations for the parameters of the renormalised Hamiltonian (which contain all powers of the spin variables) are obtained. For the model with infinite-range Gaussian interactions, the well known results for the phase transition are recovered.

1. Introduction

At present, after many years of hard work by many people, the concept of spin glasses (SG) has acquired an almost philosophical status. Although concrete experimental systems still await the answers to concrete questions, theoretical activity now embraces a wide spectrum of problems from biological (via solid state) to social. All those theories could be united rather conditionally under one common label 'Theory of spin glasses', and very often in some cases it is important to agree first on what is meant by the general label.

What I shall discuss in this paper I would prefer to specify under the term 'true spin glasses'. It is easier to explain first what is *not* implied by this term. If one has a system that at decreasing temperature undergoes a finite-order symmetry-breaking transition, it would not necessarily be an ordered system (like a ferromagnet). A trivial example is a Mattis-like magnet, where the disorder is just an irrelevant variable. However, conceptually, even much more tricky disordered systems could be referred to in the same 'class'. Intuitively these are systems that on lowering the temperature undergo the symmetry-breaking transition only once (or a finite number of times). Such systems below T_c could have many, or even infinitely many, metastable states separated by infinite barriers, and the disorder in such states could be quite relevant. However, what is important is that in the overall low-temperature range the number of separated metastable states ('valleys' in phase space) does not change, or, if talking even more intuitively, the change in their number is not crucial.

What essentially occurs in this case is that below T_c the system is becoming 'trapped' in one of many valleys in phase space, and then the situation 'inside' the valley proves to be qualitatively the same as for a ferromagnet. In other words, what is essential is that

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there are some definite states in those valleys of phase space. That is why the low-temperature state of such a system could be considered just as some (presumably rather tricky) transformation of the ferromagnetic state. Although such a scenario could in fact take place in realistic disordered statistical systems, this is not what I would refer to as 'true spin glasses'.

A quite different nature of the low-temperature state manifests itself in the Sherrington and Kirkpatrick (1975) (SK) model with infinite-range interactions studied in detail by Mezard *et al* (1984). The crucial feature of the low-temperature state of the SK model is the ultrametric (tree-like) structure of metastable states separated by infinite barriers (Mezard *et al* 1984). Of course, there could be many reasons to consider that phenomenon as just a consequence of the extremely artificial nature of the model itself. However, one could also regard it as a refined example of the general phenomenon, which manifests itself in the fractal-like structure of the free-energy landscape and which I would call the main feature of 'true spin glasses'.

On the intuitive level, this phenomenon could be described as follows. In the case of a ferromagnet, below T_c there exist two ground states separated by an infinite barrier. On increasing the temperature, the barrier 'melts' at T_c, and the two states 'merge' together to form a paramagnetic state. In the case of a 'true spin glass' well below T_c, one would have numerous metastable states separated (in the thermodynamic limit) by infinite barriers, but those barriers would have a whole spectrum of corresponding 'melting' temperatures (due to different entropy 'thicknesses'). Then, after some small increase in the temperature, the weakest barriers would melt and some families of states would merge together to form a new 'pattern' of new metastable states. Correspondingly, due to any lowering of the temperature, each valley of the phase space would 'split' into families of valleys, again separated by infinite barriers. Such a phenomenon proceeds *continuously* down to zero temperature. Therefore, there would be no definite 'ground' states corresponding to the valleys of the phase space at finite T.

Of course, the above scenario (although aesthetically attractive) is just an illustrative hypothesis without almost any hope of being proved rigorously from first principles of microscopic theory. Nevertheless, the purpose of the present paper is to show that under some assumptions a certain kind of microscopic theory could be formulated.

The principal problem of any thermodynamic theory is to define those relevant degrees of freedom that give the leading contribution to the partition function. Then no summation should be performed over all the states of the phase space.

The basic assumption of the present approach concerns the relevant states. It is assumed that all the relevant states of the spin glass could be classified in terms of the hierarchical tree. In other words, the space of relevant states is supposed to have ultrametric topology (Rammal *et al* 1986).

Consider, more specifically, the Ising spin system $\{\sigma_i\}$ $\{\sigma = \pm 1; i = 1, \ldots, N\}$. The value of the overlap between any two states $\{\sigma_i\}^{(\alpha)}$ and $\{\sigma_i\}^{(\beta)}$ could be defined as

$$q^{\alpha\beta} = \frac{1}{N}\sum_i \sigma_i^{(\alpha)}\sigma_i^{(\beta)}. \tag{1.1}$$

Overlap, in a sense, is a quantity that is the opposite of 'distance' in the space of states: close (strongly correlated) states have overlap close to 1, while 'distant' (non-correlated) states have overlap close to 0. Then, one can say that the subset $\{\sigma_i\}^{(\alpha)}(\alpha = 1, \ldots, p \ll 2^N)$ out of all the states possesses the ultrametric topology if, for any three states $\{\sigma_i\}^{(\alpha_1)}$, $\{\sigma_i\}^{(\alpha_2)}$, $\{\sigma_i\}^{(\alpha_3)}$, two out of their three overlaps $q^{\alpha_1\alpha_2}$, $q^{\alpha_1\alpha_3}$,

$q^{\alpha_2\alpha_3}$ are necessarily equal and the third one is not smaller than their value, e.g. $q^{\alpha_1\alpha_2} = q^{\alpha_1\alpha_3} \leqslant q^{\alpha_2\alpha_3}$. The space of states with such a topology is illustrated by the hierarchical tree (figure 1) in which the states are the endpoints of the tree and the overlap between any two states is determined by the number of generations from the closest common ancestor. Although looking rather peculiar, this kind of arrangement actually appears to be quite natural in the infinite-dimensional space of states (section 2).

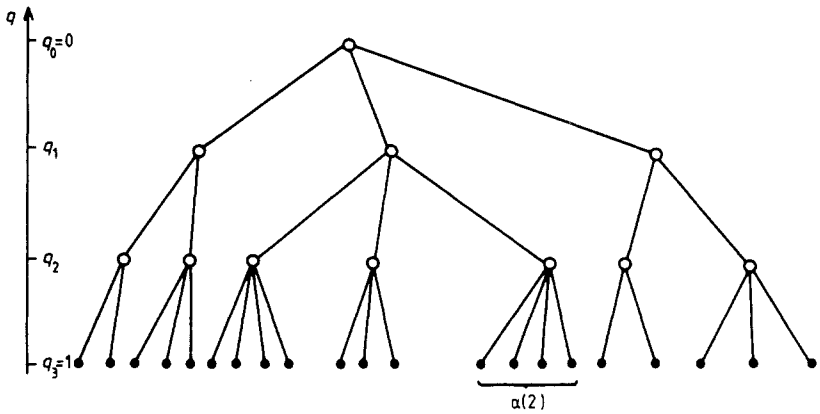


Figure 1.

Note that the above assumption does not mean that one deals with infinite-range interaction models only (although, up to now, ultrametricity has been convincingly proved only for the SK model). After fixing the ultrametric statistical properties of the relevant states, the calculations will be performed for an arbitrary quenched interaction matrix J_{ij}, and the effective Hamiltonian resulting from the renormalisation-group calculation will depend on J_{ij} as parameters.

For the calculation of the partition function over states that are classified in terms of the hierarchical tree, the renormalisation-group (RG) scheme is the most natural way of proceeding. As is conventional in any RG scheme, the calculations should be performed step by step, taking into account the whole range of the degrees of freedom from 'fast' to 'slow', which corresponds to the transition from the microscale to the macroscale. Unlike the traditional RG scheme in which the degrees of freedom are classified according to their spatial scale in ultrametric spin glasses, the degrees of freedom should be classified in terms of the scale *in phase space*. The natural measure of such a scale is given by the overlap between the states (equation (1.1)). The value of the overlaps changes from 1 to 0: if the two states are strongly correlated, i.e. they are close in phase space, their overlap is close to 1; if the two states are non-correlated, i.e. they are distant, their overlap is close to 0.

A tree-like structure implies that the states form a hierarchy of non-overlapping families. To describe it, let us introduce a regular discretisation of the interval $0 \leqslant q \leqslant 1$ by the set $\{q_l\}$, $l = 1, 2, \ldots, L \gg 1$, where $q_0 = 0$, $q_L = 1$ and $q_{l+1} - q_l = \delta q \ll 1$.

The first-level families are formed by those states that have mutual overlaps no

smaller than q_{L-1}. Such families correspond to the scale $q_{L-1} = 1 - \delta q$ and they are just one point 'above' the exact states that correspond to the scale $q_L = 1$. The families could be described by the configurations of the average site magnetisations $\{S_i(q_{L-1})\}$, where $S_i(q_{L-1}) = \langle \sigma_i \rangle_{(\alpha)}$, and the averaging at a given site goes over the states (α) belonging to one family. The 'states' described by the configurations 'live' at the scale q_{L-1} of the tree (figure 1) and they could be called the ancestor states for the exact states.

The ancestor state families at the level (scale) q_{L-2} of the tree are formed by those states that have mutual overlaps no smaller than q_{L-2}. They can be represented by the configuration of the site magnetisations $S_i(q_{L-2}) = \langle S_i(q_{L-1}) \rangle_{(\alpha)}$, where the averaging goes over the ancestor states of the previous level q_{L-1} belonging to one family (ancestor state) at the next level q_{L-2}.

Correspondingly, at some intermediate scale q_l the given ancestor state $\{S_i(q_l)\}$ is formed by averaging over its 'descendant' states of the previous scale q_{l+1}.

The basis of the RG calculations is the specially ordered way of summing up the relevant states. Let the spin system be described by some microscopic Hamiltonian $H[\sigma_i]$. At the first RG step let us fix the configurations of all the families. Then the exact spin states could be represented as $\sigma_i = S_i(q_{L-1}) + \delta S_i$, where δS_i are *small* fluctuations of the site magnetisations on the background of the ancestor state $\{S_i(q_{L-1})\}$. By definition of the ancestor states, one has $\langle \delta S \rangle = 0$ at each site, and it can also be shown that $\langle \delta S_i^2 \rangle \sim \delta q \ll 1$. Therefore the first step of the RG procedure is to sum up the elementary excitations $\{\delta S_i\}$ to get a new renormalised Hamiltonian $H_{L-1}[S_i(q_{L-1})] = H[S_i(q_{L-1})] + \delta H[S_i(q_{L-1})]$.

Correspondingly, at some intermediate scale q one has to represent the states $\{S_i(q)\}$ as small fluctuations on the background of the ancestor state $\{S_i(q - \delta q)\}$:

$$S_i(q) = S_i(q - \delta q) + \delta S_i(q). \tag{1.2}$$

Then the one-step RG transition from the Hamiltonian at scale q to that at scale $q - \delta q$ consists of summing up the fluctuations $\{\delta S_i\}$:

$$\exp\{-\beta H_{q-\delta q}[S_i]\} = \sum_{\delta S} \exp\{-\beta H_q[S_i + \delta S_i]\}. \tag{1.3}$$

Obviously the crucial point for this procedure is the statistical properties of the fluctuations δS_i, which are determined by the structure of the hierarchical tree itself. For this reason in section 2 the evolution-like generation of the tree of states is considered in general terms. It is stated that at least for a certain class of trees the fluctuations exhibit simple statistics in the 'continuous limit' $q \to 0$:

$$\langle \delta S_i \delta S_j \rangle \simeq \delta_{ij}(S_i^2/q)\delta q. \tag{1.4}$$

A direct consequence of the statistics described by (1.4) is that the effective Hamiltonian produced by the RG procedure (1.3) will contain all kinds of multiple interactions of the spins in all powers. Nevertheless, the general structure of the renormalised Hamiltonian can be derived and the RG equations for the parameters of the Hamiltonian can be obtained (section 3).

Moreover, it will be shown in section 4 that, at least for the infinite-range interaction model, the leading terms of the Hamiltonian can be summed up in a rather simple way. In the limit $q \to 0$ the Hamiltonian will be shown to be reduced to the form

$$H_q \simeq \tfrac{1}{2}\beta q \sum_{ij} \frac{S_i}{\sqrt{q}} J_{ij}^{(R)}\left(\beta, q, \frac{S^2}{q}\right) \frac{S_j}{\sqrt{q}} \tag{1.5}$$

where the renormalised 'interaction matrix' $J_{ij}^{(R)}$ depends on the temperature, the scale q and all the powers of S_i^2/q. The variables here are bounded by the condition $N^{-1}\Sigma_i S_i^2 = q$ (i.e. $\langle S_i^2/q \rangle \sim 1$). This means that the effective interactions between the effective variables $V_i = S_i/\sqrt{q}$ are equal to $qJ_{ij}^{(R)}$. Having an explicit expression for the matrix $J_{ij}^{(R)}$, we can show that at temperatures $T > 1$ the matrix has no divergence in the limit $q \to 0$. Therefore, the effective variables become decoupled, which corresponds to the paramagnetic phase.

On the other hand, at $T \leqslant 1$, matrix $J_{ij}^{(R)}$ *does* have an anomaly at a finite scale $q_c \simeq -\tau$, where $\tau = (T-1)$. It can be shown that the interactions of the eigenvectors corresponding to the edge of the eigenvalue spectrum of the initial matrix J_{ij} become divergent at $q \to q_c$. This is interpreted as the setting-up of infinite energy barriers separating the states at scale q_c. The proposed RG procedure cannot and *should not* go beyond scale q, since below T_c the actual observable thermodynamics occurs inside the separated valleys of the phase space. Such valleys correspond to the frozen ancestor states $\{S_i(q_c)\}$ of the tree.

In section 5 the prospects of treating the finite-range interaction model are briefly discussed.

2. The evolution tree of states

According to the general philosophy discussed in the introduction, the continuous process of fragmentation of the valleys of the phase space is expected to take place at decreasing temperatures below T_c. Continuing the process down to zero temperature, one could eventually come to exact spin states. Therefore, these spin states could be considered as the endpoints of the evolution branching process in phase space. It is the infinite-dimensionality of the phase space that makes it impossible for the departed branches to come close to each other again and therefore naturally produces ultrametricity.

A rather general way to 'grow up' the evolution tree of the Ising spin states is the following (see, e.g. Mezard *et al* 1987). Take the interval $[-1, +1]$ and for each site of the system independently let us start the evolution branching process:

(i) With probability $P_0(y)$ take n_1 numbers $y^{\alpha_1} \in [-1, +1]$, $\alpha_1 = 1, \ldots, n_1$.

(ii) For each y^{α_1} take n_2 numbers $y^{\alpha_1\alpha_2}$ ($\alpha_2 = 1, \ldots, n_2$) with conditional probability $P_1(y^{\alpha_1\alpha_2} | y^{\alpha_1})$.

(iii) For each $y^{\alpha_1\alpha_2}$ take n_3 numbers $y^{\alpha_1\alpha_2\alpha_3}$ ($\alpha_3 = 1, \ldots, n_3$) with conditional probability $P_2(y^{\alpha_1\alpha_2\alpha_3} | y^{\alpha_1\alpha_2})$ (for simplicity we consider the Markov processes that are known to take place in the SK model).

(iv) After L steps one will get the hierarchical tree of numbers $\{y^{\alpha_1 \cdots \alpha_L}\} \in [-1, +1]$. The tree is described by the set of probability functions

$$P_{l-1}(y^{\alpha_1 \cdots \alpha_l} | y^{\alpha_1 \cdots \alpha_{l-1}}) \tag{2.1}$$

($l = 0, 1, \ldots, L$) and is characterised by the branching numbers $(n_1, n_2, \ldots, n_L)$, which could in general be different for different levels of the tree.

(v) Finally, for the site under consideration, define the magnetisation of the Ising spin as $\sigma = \mathrm{sgn}(y^{\alpha_1 \cdots \alpha_L})$.

The above branching process has to be performed at each site of the system independently. As a result one gets the hierarchical tree of spin states

$$\{\sigma_i\}^{(\alpha_1 \cdots \alpha_L)} = \operatorname{sgn}(y^{\alpha_1 \cdots \alpha_L}). \tag{2.2}$$

The statistical properties of the tree are defined by the conditional probability functions (equation (2.1)).

Simple probabilistic considerations show that this tree is obviously ultrametric. For the overlap between two states that have a common ancestor state at level l one gets

$$\begin{aligned}\frac{1}{N}\sum_i \sigma_i^{(\alpha_1 \cdots \alpha_l \alpha_{l+1} \cdots \alpha_L)} \sigma_i^{(\alpha_1 \cdots \alpha_l \alpha'_{l+1} \cdots \alpha'_L)} \\ = \int_{-1}^{+1} \mathrm{d}y_1 \ldots \mathrm{d}y_l \, P_{l-1}(y_l|y_{l-1}) \ldots P_1(y_2|y_1) P_0(y_1) \\ \times \left(\int_{-1}^{+1} \mathrm{d}y_{l+1} \ldots \mathrm{d}y_L \, P_L(y_L|y_{L-1}) \ldots P_l(y_{l+1}|y_l) \operatorname{sgn}(y_L) \right)^2 = q_l.\end{aligned} \tag{2.3}$$

The above equation for the overlaps can also be expressed in terms of the ancestor states:

$$q_l = \frac{1}{N}\sum_i (S_i^{(\alpha_1 \cdots \alpha_l)})^2 \tag{2.4}$$

where the site magnetisations of the ancestor state are defined as

$$\begin{aligned}\langle \sigma^{(\alpha_1 \cdots \alpha_l \alpha_{l+1} \cdots \alpha_L)} \rangle_{(\alpha_{l+1} \cdots \alpha_L)} \equiv S^{(l)}(y^{\alpha_1 \cdots \alpha_l}) \\ = \int_{-1}^{+1} \mathrm{d}y_{l+1} \ldots \mathrm{d}y_L \, P_{L-1}(y_L|y_{L-1}) \ldots P_l(y_{l+1}|y^{\alpha_1 \cdots \alpha_l}) \operatorname{sgn}(y_L).\end{aligned} \tag{2.5}$$

This expression can also be rewritten in a recurrent form:

$$S^{(l)}(y) = \int_{-1}^{+1} \mathrm{d}y' \, S^{(l+1)}(y') P_l(y'|y). \tag{2.6}$$

Keeping in mind future RG calculations, the essential point in all these constructions is the statistics of the spin fluctuations:

$$\delta S^{(\alpha_1 \cdots \alpha_{l+1})} \equiv S^{\alpha_1 \cdots \alpha_l \alpha_{l+1}} - S^{\alpha_1 \cdots \alpha_l}. \tag{2.7}$$

The average square of such spin fluctuations can be expressed as

$$\begin{aligned}\langle (\delta S^{(\alpha_1 \cdots \alpha_{l+1})})^2 \rangle_{(\alpha_{l+1})} \equiv \langle (S^{(\alpha_1 \cdots \alpha_{l+1})} - S^{(\alpha_1 \cdots \alpha_l)})^2 \rangle_{(\alpha_{l+1})} \\ = \int_{-1}^{+1} \mathrm{d}y' [S^{(l+1)}(y')]^2 P_l(y'|y^{\alpha_1 \cdots \alpha_l}) \\ - \left(\int_{-1}^{+1} \mathrm{d}y' \, S^{(l+1)}(y') P_l(y'|y^{\alpha_1 \cdots \alpha_l}) \right)^2.\end{aligned} \tag{2.8}$$

At this point one has to refer to some concrete 'natural' properties of the hierarchical tree. It is natural to assume that at each step of the branching process the newly generated states are close to the state of the previous step. This means that the functions $P_l(y'|y)$,

with y taken as a parameter, have a form close to the δ-function concentrated near the value of y. Therefore there exists a small parameter of one step:

$$\delta x_l = \int \mathrm{d}\xi\, \xi^2 P_l(y+\xi|y) - \left(\int_{-1}^{+1} \mathrm{d}\xi\, \xi P_l(y+\xi|y)\right)^2 \equiv \langle \xi^2 \rangle - \langle \xi \rangle^2. \tag{2.9}$$

In terms of this parameter for the spin fluctuations (equation (2.8)) one gets

$$\langle \delta S^2 \rangle(y) = \left(\frac{\partial S^{(l)}(y)}{\partial y}\right)^2 \delta x_l. \tag{2.10}$$

On the other hand, according to equations (2.3) and (2.4) for the considered one-step transition one has

$$\delta q = q_{l+1} - q_l = \int_{-1}^{+1} \mathrm{d}y \left(\frac{\partial S^{(l)}(y)}{\partial y}\right)^2 \mathcal{P}_l(y) \delta x_l \tag{2.11}$$

where

$$\mathcal{P}_l(y) = \int_{-1}^{+1} \mathrm{d}y_1 \ldots \mathrm{d}y_{l-1}\, P_{l-1}(y|y_{l-1}) P_{l-2}(y_{l-1}|y_{l-2}) \ldots P_0(y_1). \tag{2.12}$$

In the following it will be assumed that everything is symmetric (or antisymmetric) with respect to $y \to -y$, and in the intervals $[-0, 1)$ and $(0, +1]$ all the functions are homogeneous with respect to y. Therefore δx_l do not depend on y and for the spin fluctuations one obtains

$$\langle \delta S^2 \rangle(y) = \left([\partial S^{(l)}(y)/\partial y]^2 \Big/ \int_{-1}^{+1} \mathrm{d}y'\, \mathcal{P}_l(y') [\partial S^{(l)}(y')/\partial y']^2 \right) \delta q. \tag{2.13}$$

The quantity $(\partial S/\partial y)$ has a simple meaning: it determines the number of sites in the system $\Delta N(S)$ having value of magnetisation from S to $S + \delta S$:

$$\Delta N(S) = N \mathcal{P}_l(y^{(l)}(S)) \Delta y^{(l)}(S) = N \mathcal{P}_l(y^{(l)}(S)) (\partial S^{(l)}/\partial y)^{-1} \Delta S. \tag{2.14}$$

In other words, the density of such a spin 'cluster' is

$$\Omega_l(S) = \mathcal{P}_l(y^{(l)}(S)) (\partial S^{(l)}/\partial y)^{-1}. \tag{2.15}$$

Now let us consider the behaviour of the functions under consideration in the 'continuous limit' for $q_l \to 0$, $s \to 0$ (which is expected to correspond to the region near T_c). One could consider at least two different regimes of asymptotic behaviour of the function $\Omega(S)$ for $S \to 0$.

The first case is when the function $\Omega_l(S)$ remains finite in the limit $S \to 0$. For the spin fluctuations one consequently obtains from (2.13)

$$\langle \delta S^2 \rangle \simeq \delta q. \tag{2.16}$$

This case was considered in the previous paper (Dotsenko 1987). It can be shown that it describes the 'trivial' situation corresponding to the 'spherical' model with continuous spins $|S| < qN$, and consequently more or less does not correspond to anything since the spherical model does not exhibit any hint of ultrametricity.

A quite different situation arises when the density of the spin 'cluster' $\Omega(S)$ is expected to be divergent at $S \to 0$. Since there are no reasons to expect non-analyticity, the simplest kind of divergence in the main order in S is

$$[\Omega_l(S \to 0)]^{-1} \simeq A(q_l)|S| \tag{2.17}$$

where $A(q)$ is some function of the level of the tree. In this case for the spin fluctuations (equation (2.13)) one will have

$$\langle \delta S^2 \rangle \simeq (S^2/q)\delta q. \tag{2.18}$$

Since the branching process considered goes on at each site of the system independently, one finally gets

$$\langle \delta S_i \delta S_j \rangle \simeq \delta_{ij}(S_i^2/q)\delta q. \tag{2.19}$$

Actually this is the basic *ansatz* for the RG calculations considered in the next section. The meaning of all the considerations of the present section was to show that the *ansatz* (2.18) could be a natural property of at least a certain class of tree. Although there is a great variety of different hierarchical trees, it is those (and maybe only those) that exhibit the statistics (2.18) that can be shown to have both solvable and non-trivial thermodynamics with the SG phase transitions.

3. The renormalised Hamiltonian and diagrams

We start the RG procedure with the Ising Hamiltonian

$$H_0 = \tfrac{1}{2}\beta \sum_{i \neq j} \sigma_i J_{ij} \sigma_j \tag{3.1}$$

where J_{ij} is *some* quenched interaction matrix. In the result of the renormalisation at some intermediate scale q_l, one could expect to get an effective Hamiltonian $H_{q_l}[S^{(l)}]$.

To understand the structure of the Hamiltonian, let us consider how the first few terms are generated by the one-step RG procedure (equation (1.3)). Consider what comes out of the first bilinear term that is generated by the initial Hamiltonian (3.1):

$$\exp\{-\beta H_{q_{l-1}}[S^{\alpha_1 \cdots \alpha_{l-1}}]\} = \sum_{\alpha_l} \exp\left(-\frac{\beta}{2} \sum_{i \neq j} S_i^{\alpha_1 \cdots \alpha_l} J_{ij} S_j^{\alpha_1 \cdots \alpha_l} \right). \tag{3.2}$$

To perform the averaging over the states $\{\alpha_l\}$ one has to write

$$S_i^{\alpha_1 \cdots \alpha_l} = S_i^{\alpha_1 \cdots \alpha_{l-1}} + \delta S_i^{\alpha_1 \cdots \alpha_l} \tag{3.3}$$

and the averaging over the spin fluctuations in equation (3.2) should be performed according to equation (2.19) (note also that, by definition of δS, $\langle \delta S \rangle \equiv 0$).

In addition to the initial bilinear part

$$\tfrac{1}{2}\beta \sum_{i \neq j} S_i^{\alpha_1 \cdots \alpha_{l-1}} J_{ij} S_j^{\alpha_1 \cdots \alpha_{l-1}} \tag{3.4}$$

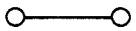
the renormalised Hamiltonian acquires an additional term

$$\ln\left[\left\langle \exp\left(-\beta \sum_{i \neq j} S_i^{\alpha_1 \cdots \alpha_{l-1}} J_{ij} \delta S_j - \tfrac{1}{2}\beta \sum_{i \neq j} \delta S_i J_{ij} \delta S_j \right) \right\rangle_{(\delta S)} \right].$$

According to equation (2.19) the part that depends on the variables S in the first order in δq is

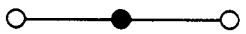
$$\tfrac{1}{2}\beta^2 \sum_{i \neq j} S_i^{\alpha_1 \cdots \alpha_{l-1}} \left(\sum_k J_{ik} \frac{(S_k^{\alpha_1 \cdots \alpha_{l-1}})^2}{q} J_{kj} \right) S_j^{\alpha_1 \cdots \alpha_{l-1}} \delta q. \tag{3.5}$$

It is convenient to represent different interactions of the Hamiltonian in terms of diagrams. The initial bilinear term can be represented as



(3.6)

where the open circles are the spins S and the line is the coupling βJ_{ij}. The term (3.5) produced by the one-step renormalisation can be represented as

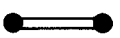


(3.7)

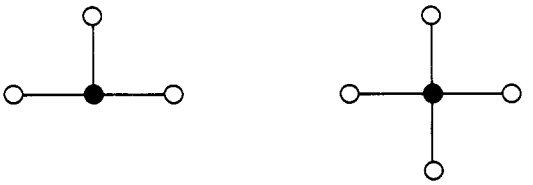
where the full circle is S^2.

The algorithm for higher-order interaction generation is obvious: one has to 'glue up' the circles in all possible ways. For example, the terms of equations (3.6) and (3.7) in the next step of the renormalisation will generate higher-order terms that can be represented as:

(3.8)



(3.9)



(3.10)

Here the full circles also represent higher-order powers of the spins. The rule is simple: the power of the spin (the order of the vertex) coincides with the number of incoming lines.

Therefore the overall Hamiltonian is composed of all kinds of such diagrams of all orders. The parameters changing with the RG change of the scale q are the weights of the vertices. The nth-order vertex can be represented as

$$(1/n)S^n G_n(q). \tag{3.11}$$

The RG equations for the vertex weights G_n can be easily obtained. In the result of the one-step renormalisation, the nth-order vertex gets one contribution from the 'gluing up' of all pairs of vertices of the kth and $(n-k)$th order $(k = 1, 2, \ldots, n-1)$ and another from 'self-action'. The resulting RG equations are

$$\frac{\mathrm{d}}{\mathrm{d}q} G_n(q) = \frac{1}{q}\left(n \sum_{k=1}^{n-1} G_k G_{n-k} - \frac{n(n-1)}{2} G_n\right) \tag{3.12}$$

where $G_1 = 1$.

Strictly speaking the problem of solving equations (3.12) is not self-consistent since the region where the initial conditions $G_{n\geqslant 2}(q = 1) = 0$ are defined is beyond the region $q \ll 1$ where the equations are valid. However, it can be proved that equations (3.12) have a 'universal' asymptotic behaviour for $q \to 0$, which does not depend on the initial

conditions. After some algebra, one can easily check that in the main order in q the solutions are

$$G_{2k} \simeq [(-2)^k/q^k](1 - k^2 q) \tag{3.13}$$

$$G_{2k+1}(q) \simeq [(-2)^k/q^k](2k + 1). \tag{3.14}$$

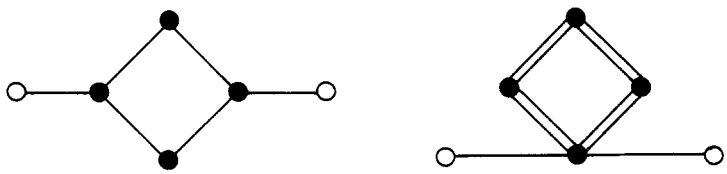
Note that according to equations (3.13) and (3.14), the even vertices $S^{2k}G_{2k}(q) \sim 1$, while the odd vertices

$$S^{2k+1}G_{2k+1}(q) \sim S \sim \sqrt{q} \ll 1. \tag{3.15}$$

4. The phase transition

Now we face the problem of summing up all the diagrams constituting the Hamiltonian with vertices given by equations (3.13) and (3.14). Of course, this is a hard problem to solve for a general matrix, but at least for the model with infinite-range interactions it can be solved in a rather simple way.

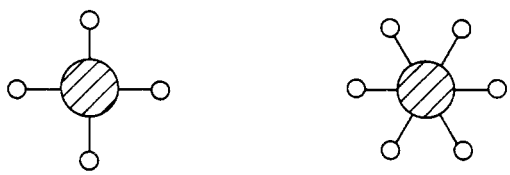
The main simplification is due to the fact that, in the main order in N^{-1} ($N \to \infty$), one can ignore all diagrams containing loops. Examples of diagrams containing loops are



(4.1)

Here the lines I_{ij} are described by the Gaussian distribution with $\overline{J_{ij}^2} = J_0^2/N$. One can easily estimate that any loop produces an additional factor N^{-1}. Therefore, in order to obtain the expression for the renormalised Hamiltonian, one has to sum up only tree-like diagrams.

The second step is to classify all the diagrams according to the number of linear-power spins contained in them. The crucial point is that in the limit $q \to 0$ (near T_c) all the diagrams that contain an additional pair of linear spins acquire a small additional factor proportional to q. For that reason, in the main order in q, all the diagrams that contain more than two linear spins, such as

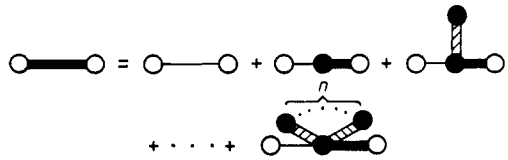


(4.2)

can be ignored. For the same reason, one should not take into account diagrams that contain odd-order vertices (equations (3.14) and (3.15)).

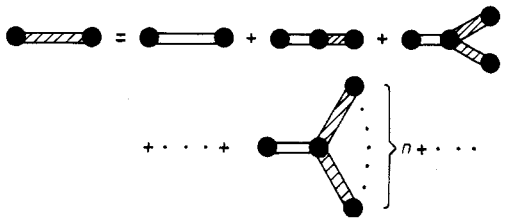
Therefore, in the main order in q and N^{-1}, one has to sum up all the tree-like diagrams containing only even vertices and with only two end linear spins. The sum of such diagrams could be considered as the generalised bilinear interaction $S_i J_{ij}^{(R)}(\beta, q, S^2)S_j$

and can be represented in terms of the equation



(4.3)

Here the thick full line represents $J_{ij}^{(R)}$ and the hatched line represents the sum of all the tree-like diagrams in which all the interactions are squared:



(4.4)

One can easily check that all other diagrams with higher powers of the interactions, such as

are of higher orders in N^{-1}.

In explicit form, equations (4.3) and (4.4) can be written as

$$J_{ij}^{(R)} = J_{ij} + \beta \sum_{l=1}^{N} J_{il}\left(\sum_{n=1}^{\infty} S_l^{2n} \frac{G_{2n}(q)}{2} I_l^{n-1}(q)\right) J_{lj}^{(R)} \tag{4.5}$$

where

$$I_i(q) = -\beta^2 \sum_{j=1}^{N} J_{ij}^2 \left(\sum_{n=1}^{\infty} S_j^{2n} \frac{G_{2n}(q)}{2} I_j^{n-1}(q)\right). \tag{4.6}$$

Using equation (3.13), one can obtain the solution of equation (4.5)

$$\hat{J}^{(R)} = \hat{J}(1 + \beta \hat{J}')^{-1} \tag{4.7}$$

where

$$J'_{ij} = J_{ij} \kappa_j(\beta, q, S^2) \tag{4.8}$$

and

$$\kappa_j = \frac{S_j^2}{q}\left(1 + 2\frac{S_j^2}{q} I_j\right)^{-1}. \tag{4.9}$$

From equation (4.6) the equations for the I can be obtained as

$$I_i = \beta^2 \sum_j J_{ij}^2 \frac{S_j^2}{q}\left(1 + 2\frac{S_j^2}{q} I_j\right)^{-1}. \tag{4.10}$$

Therefore in the limit $q \to 0$ we have the asymptotic theory that is described by the free energy

$$\mathcal{F}_{q\to 0} \simeq \tfrac{1}{2}\beta q \sum_{ij} V_i J_{ij}^{(R)} V_j \tag{4.11}$$

where the effective variables

$$V_i \equiv S_i / \sqrt{q} \tag{4.12}$$

are bounded by the condition

$$\frac{1}{N}\sum_i V_i^2 = 1. \tag{4.13}$$

The asymptotic expression (equation (4.11)) means that the effective interactions between the variables V_i are

$$\beta q J_{ij}^{(R)}. \tag{4.14}$$

In the limit $q \to 0$ one has two possibilities. The first, a 'trivial' one, is when there are no anomalies (divergences) in the interaction matrix $\hat{J}^{(R)}$. In this case the variables V_i become decoupled (the effective temperature goes to infinity for $q \to 0$) and consequently the corresponding state of the system is paramagnetic.

However, another possibility exists. It will be shown that at $T < J_0$ the matrix $\hat{J}^{(R)}$ has an anomaly. Namely, the interactions of the eigenvectors corresponding to the edge of the eigenvalue spectrum of the initial matrix $\hat{J}$ become divergent. This means that in such case the limit $q \to 0$ cannot be reached. Consider equation (4.10). Assuming self-averaging for the quantity $\langle I \rangle \equiv (1/N)\Sigma_i I_i$ one gets

$$\langle I \rangle = \tfrac{1}{4}[(1 + 8\beta^2 J_0^2)^{1/2} - 1] \simeq \tfrac{1}{2} - \tfrac{2}{3}\tau \tag{4.15}$$

where

$$\beta J_0 = 1 - \tau \qquad \tau = (T - J_0)/J_0 \ll 1. \tag{4.16}$$

Correspondingly

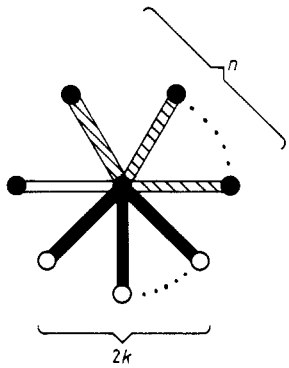
$$\beta\langle \kappa \rangle \simeq (1/2J_0)\,(1 - \tfrac{1}{3}\tau). \tag{4.17}$$

Since the spectrum of the eigenvalues of the initial interaction matrix J_{ij} is in the interval $(-2J_0, +2J_0)$, the spectrum of the effective interaction matrix (4.7) begins at

$$-2J_0/(1 - 2J_0\beta\langle \kappa \rangle). \tag{4.18}$$

The result (4.17) shows that this value becomes divergent at $\tau \to +0$ (when $T \to T_c = J_0$ from above). This can be interpreted as the setting up of infinite barriers at T_c, which would divide the phase space into valleys below T_c.

The above speculations indicate that for $T < T_c$ the limit $q \to 0$ cannot be reached and the RG procedure should stop at some $q(\tau) \neq 0$. To show that, we have to perform the above calculations for $J_{ij}^{(R)}$ in the next order in q. To do that we have to take into account the additional class of diagrams that contain an even number of additional 'branches' $\hat{J}^{(R)}$ such as



(4.19)

This gives a non-zero contribution when the end linear spins of the 'branches' $\hat{J}^{(R)}$ coincide in pairs. For the diagram (4.19) it gives the additional multiplier

$$\tfrac{1}{2}\sum_j (J_{ij}^{(R)})^2 S_j^2 \tag{4.20}$$

where the values of $J_{ij}^{(R)}$ should be taken at $q \to 0$. Assuming self-averaging, expression (4.20) is reduced to

$$\tfrac{1}{2}q\langle (J_{ij}^{(R)})^2\rangle. \tag{4.21}$$

Using the expression for $\hat{J}^{(R)}$ (equation (4.7)) at the phase transition point, after some algebra one can easily check that

$$\langle (J_{ij}^{(R)})^2\rangle = \tfrac{1}{2}. \tag{4.22}$$

Therefore, instead of equation (4.10), after summing up all the diagrams (4.19) for the parameters I one finally obtains the equation

$$I_i = \beta^2 \sum_j J_{ij}^2 \frac{S_j^2}{q}\left(1 + 2\frac{S_j^2}{q} I_j\right)^{-1}(1 + \tfrac{1}{2}q) \equiv \beta^2 \sum_j J_{ij}^2 \kappa_j. \tag{4.23}$$

Note that in obtaining equation (4.23) one should use the expressions for the vertices $G_{2k}(q)$ in the next order in q, too (equation (3.13)). However, one can easily check that the first correction $k^2 q$ in $G_{2k}(q)$ appears to be 'marginal', giving no contribution to equation (4.23).

According to equation (4.18) the divergence of the spectrum of the matrix $\hat{J}^{(R)}$ first appears when

$$\beta\langle\kappa\rangle = 1/(2J_0) \tag{4.24}$$

or according to equation (4.24), when

$$\langle I\rangle = \tfrac{1}{2}\beta J_0. \tag{4.25}$$

This is the condition for the boundary for the RG procedure value of q.

The equation for $\langle I\rangle$ can be obtained from equation (4.23):

$$\langle I\rangle = \beta^2 J_0^2 (1 + 2\langle I\rangle)^{-1}(1 + \tfrac{1}{2}q)^{-1}. \tag{4.26}$$

Solving it with $\langle I\rangle$ given by equation (4.25), one finally obtains that the boundary value of q at which the divergence in the interactions occurs is

$$q(\tau) \simeq -\tau \tag{4.27}$$

where $\tau = T/J_0 - 1$ ($|\tau| \ll 1$).

The value of $q(\tau)$ can be interpreted as the scale in the phase space at which the infinite barriers grow. In other words, at $T < T_c = J_0$ the system should be expected to be trapped in one of the valleys of the phase space, which corresponds to one of the ancestor states of the ultrametric tree characterised by the value of the average square magnetisations

$$\frac{1}{N}\sum_{i=1}^{N} S_i^2 = q(\tau). \tag{4.28}$$

5. Conclusions

In the present paper a new RG method has been proposed, which in principle could provide a thermodynamic description for any spin glass (or any disordered system) with an ultrametric arrangement of thermodynamically relevant states. Explicit calculations have been performed for a model with Gaussian infinite-range interactions, and the well known results for the phase transition have been recovered.

The present RG method is proposed, of course, not just as an additional way to solve the SK model (it is the right time to give up solving the SK model). There is hope that the method could have much wider applications. The reasons for this are as follows. The method itself is based 'only' on the assumption of the ultrametric structure of the relevant states of the disordered system (which could be hoped to be a quite natural property), while the microstructure of the ultrametricity could be described by some rather general evolution functions $P_q(y|y')$ (equation (2.1)).

The RG equation for the parameters of the effective Hamiltonian and even the general structure of the effective Hamiltonian itself can be obtained in terms of the *arbitrary* spin–spin interaction matrix J_{ij}. In the case of infinite-range J_{ij}, the derivation of the renormalised Hamiltonian near T_c was shown to be a not very difficult problem. The reason for that is obvious: the corresponding diagrams have no loops. On the other hand, for the derivation of the renormalised Hamiltonian of the finite-range interaction model, certain types of loops must be taken into account.

The principal question that remains to be solved is to what extent the problem of taking into account the loops is only a technical problem. As for the main question, if the basic ultrametric hypothesis is correct, it could be answered only *a posteriori*.

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